MBS-003-1164001 Seat No. ________

M. Sc. (Sem. IV) (CBCS) Examination

April / May - 2018

Mathematics : MATH.CMT - 4001

(Linear Algebra)

(New Course)

Faculty Code : 003

Subject Code : 1164001

Time : $2\frac{1}{2}$ Hours] [Total Marks : **70**

Instructions : (1) Answer all the questions

(2) Each question carries 14 marks.

(3) Vector spaces considered here are finite-dimensional

1 Answer any **seven** : **7×2=14**

(a) When is an element $T \in A_F(V)$ said to be invertible? If $T \in A_{\mathbb{R}}\left(\mathbb{R}^{(7)}\right)$ is invertible, then find $r(T)$.

(b) Why does there exist no vector space V over $\mathbb{R}$ such that $\dim_{\mathbb{R}} A_{\mathbb{R}}(V) = 85$?

(c) Let $T \in A_F(V)$ and let $p(x)$ be the minimal polynomial of T over F. If $\lambda \in F$ is a characteristic root of T, then show that $p(\lambda) = 0$.

(d) When are $T, S \in A_F(V)$ said to be similar?

(e) Let $T : \mathbb{Q}^{(3)} \to \mathbb{Q}^{(3)}$ be defined by $T(1,0,0) = (0,1,0)$, $T(0,1,0) = (0,0,1), T(0,0,1) = (0,0,0)$ and extend T linearly to the whole of $\mathbb{Q}^{(3)}$. Verify that T is nilpotent and find the index of nilpotence of T.

(f) Let $A \in \mathbb{R}_5$. When is A said to be a basic Jordan block belonging to $\sqrt{13}$?

(g) State Cramer's rule.

(h) Let $(V, <,>)$ be an inner product space over $\mathbb{C}$. Let $N \in A_{\mathbb{C}}(V)$ be normal. If u $v \in KerN$, then show that $v \in Ker\, N^*$.

(i) Let $(V, <,>)$ be as in (h). If $T \in A_{\mathbb{C}}(V)$ is Hermitian, then show that $< T(v), v > \in \mathbb{R}$ for any $v \in V$.

(j) State the polarization identity.

2 Answer any **Two** : **7×2=14**

(a) (i) Let $T \in A_F(V)$. Prove that T satisfies a nontrivial polynomial $q(x) \in F[x]$.

(ii) If $T \in A_F(V)$ is invertible, then show that T^{-1} is a polynomial expression in T over F.

(b) If V is a n–dimensional vector space over a field F, then prove that $A_F(V)$ and F_n are isomorphic as algebras over F.

(c) Let $T, S \in A_F(V)$. If S is regular, then show that T and STS^{-1} have the same minimal polynomial.

3 (a) If n_1 is the index of nilpotence of a nilpotent $T \in A_F(V)$ and if $v \in V$ is such that $T^{n_1 - 1}(v) \neq 0$, then prove that $\{v, T(v) .....T^{n_1 - 1}(v)\}$ is linearly independent over F. **5**

(b) Let $V = V_1 \oplus V_2$, where V_1 and V_2 are invariant under $T \in A_F(V)$. If $p_i(x) \in F[x]$ is the minimal polynomial of $T|V_i$ for each $i \in \{1, 2\}$, then show that the minimal polynomial of T over F is the least common multiple of $p_1(x)$ and $p_2(x)$. **5**

(c) Let $A, B \in F_n$. Show that $tr(AB) = tr(BA)$. **4**

OR

3 (a) Let $T, S \in A_F(V)$ be similar. Show that given a basis B_1 of V over F, there exists a basis B_2 of V over F such that the matrix of T in B_1 equals the matrix of S in B_2. **5**

(b) Prove that any $T \in A_F(V)$ satisfies its characteristic polynomial. **5**

(c) Let $A \in \mathbb{C}_n$ be Hermitian. Show that any characteristic root of A is real. **4**

4 Answer any **Two** : **7×2=14**

(a) If $\dim_F(V) = n$ and if $T \in A_F(V)$ has all its characteristic roots in F, then prove that T satisfies a polynomial of degree n over F.

(b) Let $A \in F_n$. Show that $\det(A) = \det(A')$.

(c) Let $A \in F_n$ and suppose that K is the splitting field of the minimal polynomial of A over F. Show that there is an invertible matrix $C \in K_n$ such that CAC^{-1} is in Jordan form.

5 Answer any **Two** : **7×2=14**

(a) Let $T \in A_F(V)$. If V is cyclic relative to T, then prove that there exists a basis B of V over F such that the matrix of T in B is $C(p(x))$, where $p(x)$ is the minimal polynomial of T over F.

(b) Let $(V, <,>)$ be an inner product space over $\mathbb{C}$. Let $T \in A_{\mathbb{C}}(V)$. Show that T is unitary if and only if it takes an orthonormal basis of V into an orthonormal basis of V.

(c) Let V be a vector space over $\mathbb{R}$ and let f be a symmetric bilinear form on V. Prove that there is a basis B of V that the matrix of f in B is diagonal.

(d) Let $n \geq 1$. Show that the mapping $f : F_n \to F_n$ defined by $f(A) = A'$ is an adjoint of F_n.